Theoretical Question 1: *Ping-Pong Resistor*

1. Answers

(a) $F_R = -\frac{1}{2}\pi R^2 \varepsilon_0 \frac{V^2}{d^2}$

(b) $\chi = -\varepsilon_0 \frac{\pi r^2}{d}$

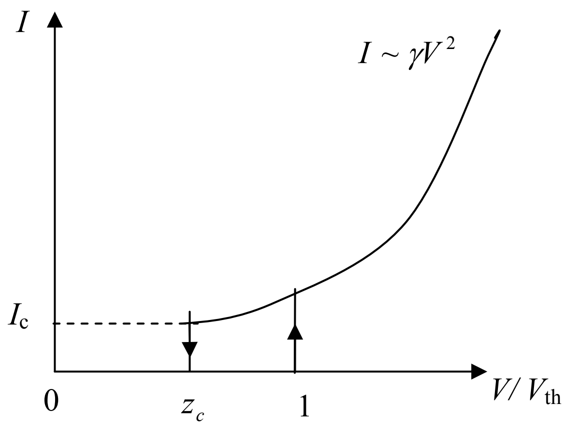
(c) $V_{th} = \sqrt{\frac{2mgd}{\chi}}$

(d) $v_s = \sqrt{\alpha V^2 + \beta}$

$$\alpha = \left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi}{m}\right), \quad \beta = \left(\frac{\eta^2}{1+\eta^2}\right)(2gd)$$

(e) $\gamma = \sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi^3}{2md^2}}$

(f) $V_c = \sqrt{\frac{1-\eta^2}{1+\eta^2}}\sqrt{\frac{mgd}{\chi}}$, $I_c = \frac{2\eta\sqrt{1-\eta^2}}{(1+\eta)(1+\eta^2)} g\sqrt{m\chi}$



2. Solutions

(a) [1.2 points]

The charge Q induced by the external bias voltage V can be obtained by applying the Gauss law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = Q \quad \text{(a1)}$$

$$Q = \varepsilon_0 E \cdot (\pi R^2) = \varepsilon_0 \left(\frac{V}{d} \right) \cdot (\pi R^2), \quad \text{(a2)}$$

where $V = Ed$.

The energy stored in the capacitor:

$$U = \int_0^V Q(V')dV' = \int_0^V \varepsilon_0 \pi R^2 \left(\frac{V'}{d} \right) dV' = \frac{1}{2} \varepsilon_0 \pi R^2 \frac{V^2}{d} . \quad \text{(a3)}$$

The force acting on the plate, when the bias voltage V is kept constant:

$$\therefore F_{\mathrm{R}} = + \frac{\partial U}{\partial d} = - \frac{1}{2} \varepsilon_0 \pi R^2 \frac{V^2}{d^2} . \quad \text{(a4)}$$

<u>[An alternative solution:]</u>

Since the electric field E' acting on one plate should be generated by the other plate and its magnitude is

$$E' = \frac{1}{2} E = \frac{V}{2d}, \quad \text{(a5)}$$

the force acting on the plate can be obtained by

$$F_{\mathrm{R}} = QE' . \quad \text{(a6)}$$

(b) [0.8 points]

The charge q on the small disk can also be calculated by applying the Gauss law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = q . \quad \text{(b1)}$$

Since one side of the small disk is in contact with the plate,

$$q = -\varepsilon_0 E \cdot (\pi r^2) = -\varepsilon_0 \frac{\pi r^2}{d} V = \chi V . \quad \text{(b2)}$$

Alternatively, one may use the area ratio for $q = -\left(\frac{\pi r^2}{\pi R^2}\right)Q$.

$$\therefore \chi = -\varepsilon_0 \frac{\pi r^2}{d}. \tag{b3}$$

(c) [0.5 points]

The net force, F_{net}, acting on the small disk should be a sum of the gravitational and electrostatic forces:

$$F_{\text{net}} = F_{\text{g}} + F_{\text{e}}. \tag{c1}$$

The gravitational force: $F_{\text{g}} = -mg$.

The electrostatic force can be derived from the result of (a) above:

$$F_{\text{e}} = \frac{1}{2}\varepsilon_0 \frac{\pi r^2}{d^2} V^2 = \frac{\chi}{2d} V^2. \tag{c2}$$

In order for the disk to be lifted, one requires $F_{\text{net}} > 0$:

$$\frac{\chi}{2d} V^2 - mg > 0. \tag{c3}$$

$$\therefore V_{\text{th}} = \sqrt{\frac{2mgd}{\chi}}. \tag{c4}$$

(d) [2.3 points]

Let v_{s} be the steady velocity of the small disk just after its collision with the bottom plate. Then the *steady-state* kinetic energy K_{s} of the disk just above the bottom plate is given by

$$K_{\text{s}} = \frac{1}{2} m \text{v}_{\text{s}}^2. \tag{d1}$$

For each round trip, the disk gains electrostatic energy by

$$\Delta U = 2qV. \tag{d2}$$

For each inelastic collision, the disk lose its kinetic energy by

$$\Delta K_{\text{loss}} = K_{\text{before}} - K_{\text{after}} = (1-\eta^2) K_{\text{before}} = \left(\frac{1}{\eta^2} - 1\right) K_{\text{after}}. \tag{d3}$$

Since K_{s} is the energy after the collision at the bottom plate and $(K_{\text{s}} + qV - mgd)$ is the energy before the collision at the top plate, the total energy loss during the round trip can be written in terms of K_s:

$$\Delta K_{\text{tot}} = \left(\frac{1}{\eta^2} - 1\right) K_s + (1-\eta^2)(K_s + qV - mgd). \tag{d4}$$

In its steady state, ΔU should be compensated by ΔK_{tot}.

$$2qV = \left(\frac{1}{\eta^2} - 1\right) K_s + (1-\eta^2)(K_s + qV - mgd). \tag{d5}$$

Rearranging Eq. (d5), we have

$$\begin{aligned} K_s &= \frac{\eta^2}{1-\eta^4}\left[(1+\eta^2)qV + (1-\eta^2)mgd\right] \\ &= \left(\frac{\eta^2}{1-\eta^2}\right) qV + \left(\frac{\eta^2}{1+\eta^2}\right) mgd \\ &= \frac{1}{2} m \mathrm{v}_s^2. \end{aligned} \tag{d6}$$

Therefore,

$$\mathrm{v}_s = \sqrt{\left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi V^2}{m}\right) + \left(\frac{\eta^2}{1+\eta^2}\right)(2gd)}. \tag{d7}$$

Comparing with the form:

$$\mathrm{v}_s = \sqrt{\alpha V^2 + \beta}, \tag{d8}$$

$$\alpha = \left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi}{m}\right), \quad \beta = \left(\frac{\eta^2}{1+\eta^2}\right)(2gd). \tag{d9}$$

[An alternative solution:]

Let v_n be the velocity of the small disk just after n-th collision with the bottom plate. Then the kinetic energy of the disk just above the bottom plate is given by

$$K_n = \frac{1}{2} m \mathrm{v}_n^2. \tag{d10}$$

When it reaches the top plate, the disk gains energy by the increase of potential energy:

$$\Delta U_{\text{up}} = qV - mgd. \tag{d11}$$

Thus, the kinetic energy just before its collision with the top plate becomes

$$K_{n-\text{up}} = \frac{1}{2} m \mathrm{v}_{\text{up}}^2 = K_n + \Delta U_{\text{up}}. \tag{d12}$$

Since $\eta = \mathrm{v}_{\mathrm{after}} / \mathrm{v}_{\mathrm{before}}$, the kinetic energy after the collision with the top plate becomes scaled down by a factor of η^2:

$$K'_{n-\mathrm{up}} = \eta^2 \cdot K_{n-\mathrm{up}} . \quad \text{(d13)}$$

Now the potential energy gain by the downward motion is:

$$\Delta U_{\mathrm{down}} = qV + mgd \quad \text{(d14)}$$

so that the kinetic energy just before it collides with the bottom plate becomes:

$$K_{n-\mathrm{down}} = K'_{n-\mathrm{up}} + \Delta U_{\mathrm{down}} . \quad \text{(d15)}$$

Again, due to the loss of energy by the collision with the bottom plate, the kinetic energy after its $(n+1)$-th collision can be obtained by

$$\begin{aligned} K_{n+1} &= \eta^2 \cdot K_{n-\mathrm{down}} \\ &= \eta^2 (K'_{n-\mathrm{up}} + \Delta U_{\mathrm{down}}) \\ &= \eta^2 (\eta^2 (K_n + \Delta U_{\mathrm{up}}) + \Delta U_{\mathrm{down}}) \\ &= \eta^2 (\eta^2 (K_n + qV - mgd) + qV + mgd) \\ &= \eta^4 K_n + \eta^2 (1+\eta^2) qV + \eta^2 (1-\eta^2) mgd. \end{aligned} \quad \text{(d16)}$$

As $n \to \infty$, we expect the velocity $\mathrm{v}_n \to \mathrm{v}_s$, that is, $K_n \to K_s = \frac{1}{2} m \mathrm{v}_s^2$:

$$\begin{aligned} K_s &= \frac{1}{1-\eta^4} \left[\eta^2 (1+\eta^2) qV + \eta^2 (1-\eta^2) mgd \right] \\ &= \left(\frac{\eta^2}{1-\eta^2} \right) qV + \left(\frac{\eta^2}{1+\eta^2} \right) mgd \\ &= \frac{1}{2} m \mathrm{v}_s^2 \end{aligned} \quad \text{(d17)}$$

(e) [2.2 points]

The amount of charge carried by the disk during its round trip between the plates is $\Delta Q = 2q$, and the time interval $\Delta t = t_+ + t_-$, where t_+ (t_-) is the time spent during the up- (down-) ward motion respectively.

Here t_+ (t_-) can be determined by

$$\begin{aligned} \mathrm{v}_{0+} t_+ + \frac{1}{2} a_+ t_+^2 &= d \\ \mathrm{v}_{0-} t_- + \frac{1}{2} a_- t_-^2 &= d \end{aligned} \quad \text{(e1)}$$

where v_{0+} (v_{0-}) is the initial velocity at the bottom (top) plate and a_+ (a_-) is the up-

(down-) ward acceleration respectively.

Since the force acting on the disk is given by

$$F = ma_{\pm} = qE \mp mg = \frac{qV}{d} \mp mg \,, \tag{e2}$$

in the limit of $mgd << qV$, $a_{\pm}$ can be approximated by

$$a_0 = a_+ = a_- \approx \frac{qV}{md} \,, \tag{e3}$$

which implies that the upward and down-ward motion should be symmetric. Thus, Eq.(e1) can be described by a single equation with $t_0 = t_+ = t_-$, $\mathrm{v_s} = \mathrm{v}_{0+} = \mathrm{v}_{0-}$, and $a_0 = a_+ = a_-$. Moreover, since the speed of the disk just after the collision should be the same for the top- and bottom-plates, one can deduce the relation:

$$\mathrm{v_s} = \eta(\mathrm{v_s} + a_0 t_0) \,, \tag{e4}$$

from which we obtain the time interval $\Delta t = 2t_0$,

$$\Delta t = 2t_0 = 2\left(\frac{1-\eta}{\eta}\right)\frac{\mathrm{v_s}}{a_0} \,. \tag{e5}$$

From Eq. (d6), in the limit of $mgd << qV$, we have

$$K_s = \frac{1}{2}m\mathrm{v_s^2} \approx \left(\frac{\eta^2}{1-\eta^2}\right)qV \,. \tag{e6}$$

By substituting the results of Eqs. (e3) and (e6), we get

$$\Delta t = 2\left(\frac{1-\eta}{\eta}\right)\sqrt{\frac{2\eta^2}{1-\eta^2}}\sqrt{\frac{md^2}{qV}} = 2\sqrt{\frac{1-\eta}{1+\eta}}\sqrt{\frac{2md^2}{\chi V^2}} \,. \tag{e7}$$

Therefore, from $I = \dfrac{\Delta Q}{\Delta t} = \dfrac{2q}{\Delta t}$,

$$I = \frac{2q}{\Delta t} = \chi V\sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi V^2}{2md^2}} = \sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi^3}{2md^2}}V^2 \,. \tag{e8}$$

$$\therefore \gamma = \sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi^3}{2md^2}} \tag{e9}$$

<u>[Alternative solution #1:]</u>

Starting from Eq. (e3), we can solve the quadratic equation of Eq. (e1) so that

$$t_{\pm} = \frac{v_{0\pm}}{a_0}\left(\sqrt{1+\frac{2da_0}{v_{0\pm}^2}}-1\right). \quad \text{(e10)}$$

When it reaches the steady state, the initial velocities $v_{0\pm}$ are given by

$$v_{0+} = v_s \quad \text{(e11)}$$

$$v_{0-} = \eta\cdot(v_s + a_0 t_+) = \eta v_s\sqrt{1+\frac{2da_0}{v_s^2}}, \quad \text{(e12)}$$

where v_s can be rewritten by using the result of Eq. (e6),

$$v_s^2 \approx \alpha V = \left(\frac{\eta^2}{1-\eta^2}\right)\frac{2qV}{m} = \left(\frac{\eta^2}{1-\eta^2}\right)2a_0 d. \quad \text{(e13)}$$

As a result, we get $v_{0-} \cong \eta v_s \cdot \frac{1}{\eta} = v_s$ and consequently $t_{\pm} = \frac{v_s}{a_0}\left(\frac{1}{\eta}-1\right)$, which is equivalent to Eq. (e4).

[Alternative solution #2:]

The current I can be obtained from

$$I = \frac{2q}{\Delta t} = \frac{2q\bar{v}}{d}, \quad \text{(e14)}$$

where $\bar{v}$ is an average velocity. Since the up and down motions are symmetric with the same constant acceleration in the limit of $mgd << qV$,

$$\bar{v} = \frac{1}{2}\left(v_s + \frac{v_s}{\eta}\right). \quad \text{(e15)}$$

Thus, we have

$$I = \frac{q}{2d}\left(1+\frac{1}{\eta}\right)v_s. \quad \text{(e16)}$$

Inserting the expression (Eq. (e15)) of v_s into Eq. (e16), one obtains an expression identical to Eq. (e8).

(f) [3 points]

The disk will lose its kinetic energy and eventually cease to move when the disk can not reach the top plate. In other words, the threshold voltage V_c can be determined from the condition that the velocity v_{0-} of the disk at the top plate is zero, i.e., $v_{0-} = 0$.

In order for the disk to have $v_{0-} = 0$ at the top plate, the kinetic energy $\overline{K}_s$ at the top plate should satisfy the relation:

$$\overline{K}_s = K_s + qV_c - mgd = 0\,, \qquad \text{(f1)}$$

where K_s is the *steady-state* kinetic energy at the bottom plate after the collision. Therefore, we have

$$\left(\frac{\eta^2}{1-\eta^2}\right)qV_c + \left(\frac{\eta^2}{1+\eta^2}\right)mgd + qV_c - mgd = 0\,, \qquad \text{(f2)}$$

or equivalently,

$$(1+\eta^2)qV_c - (1-\eta^2)mgd = 0\,. \qquad \text{(f3)}$$

$$\therefore \quad qV_c = \frac{1-\eta^2}{1+\eta^2}mgd \qquad \text{(f4)}$$

From the relation $q = \chi V_c$,

$$\therefore V_c = \sqrt{\frac{1-\eta^2}{1+\eta^2}}\sqrt{\frac{mgd}{\chi}}\,. \qquad \text{(f5)}$$

In comparison with the threshold voltage V_{th} of Eq. (c4), we can rewrite Eq. (f5) by

$$V_c = z_c V_{th} \qquad \text{(f6)}$$

where z_c should be used in the plot of I vs. (V/V_{th}) and

$$z_c = \sqrt{\frac{1-\eta^2}{2(1+\eta^2)}}\,. \qquad \text{(f7)}$$

[Note that an alternative derivation of Eq. (f1) is possible if one applies the energy compensation condition of Eq. (d5) or the recursion relation of Eq. (d17) at the top plate instead of the bottom plate.]

Now we can setup equations to determine the time interval $\Delta t = t_- + t_+$:

$$v_{0-}t_- + \frac{1}{2}a_- t_-^2 = d \qquad \text{(f8)}$$

$$v_{0+}t_+ + \frac{1}{2}a_+ t_+^2 = d \qquad \text{(f9)}$$

where the accelerations are given by

$$a_+ = \frac{qV_c}{md} - g = \left[\frac{1-\eta^2}{1+\eta^2} - 1\right]g = \left(\frac{-2\eta^2}{1+\eta^2}\right)g \qquad \text{(f10)}$$

$$a_- = \frac{qV_c}{md} + g = \left[\frac{1-\eta^2}{1+\eta^2} + 1\right] g = \left(\frac{2}{1+\eta^2}\right) g \quad \text{(f11)}$$

$$\frac{a_+}{a_-} = -\eta^2 \quad \text{(f12)}$$

Since $v_{0-} = 0$, we have $v_{0+} = \eta(a_- t_-)$ and $t_-^2 = 2d / a_-$.

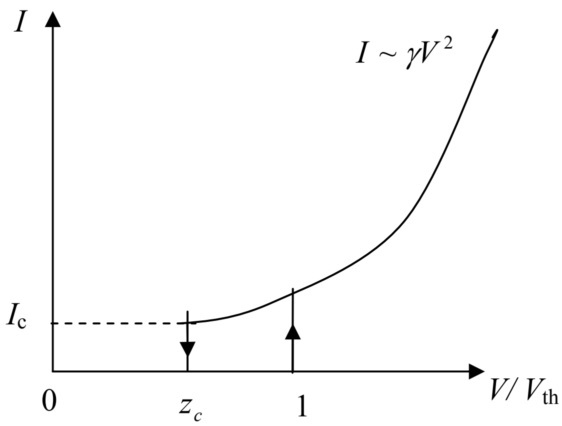
$$t_- = \sqrt{\frac{2d}{a_-}} = \sqrt{(1+\eta^2)\left(\frac{d}{g}\right)}, \quad \text{(f13)}$$

By using $v_{0+}^2 = \eta^2(2da_-) = -2da_+$, we can solve the quadratic equation of Eq. (f9):

$$t_+ = \frac{v_{0+}}{a_+}\left(\sqrt{1+\frac{2da_+}{v_{0+}^2}} - 1\right) = -\frac{v_{0+}}{a_+} = \sqrt{\frac{2d}{|a_+|}} = \sqrt{\left(\frac{1+\eta^2}{\eta^2}\right)\left(\frac{d}{g}\right)} = \frac{t_-}{\eta}. \quad \text{(f14)}$$

$$\therefore \Delta t = t_- + t_+ = \left(1+\frac{1}{\eta}\right)\sqrt{\left(1+\eta^2\right)\left(\frac{d}{g}\right)} \quad \text{(f15)}$$

$$I_c = \frac{\Delta Q_c}{\Delta t} = \frac{2q}{\Delta t} = \frac{2\chi V_c}{\Delta t} = \frac{2\eta\sqrt{1-\eta^2}}{(1+\eta)(1+\eta^2)} g\sqrt{m\chi}. \quad \text{(f16)}$$



[A more elaborate Solution:]

One may find a general solution for an arbitrary value of V. By solving the quadratic equations of Eqs. (f8) and (f9), we have

$$t_{\pm} = \frac{\mathrm{v}_{0\pm}}{a_{\pm}}\left[-1+\sqrt{1+\frac{2da_{\pm}}{\mathrm{v}_{0\pm}^2}}\right]. \qquad \text{(f17)}$$

(It is noted that one has to keep the smaller positive root.)

To simplify the notation, we introduce a few variables:

(i) $y = \dfrac{V}{V_{\text{th}}}$ where $V_{\text{th}} = \sqrt{\dfrac{2mgd}{\chi}}$,

(ii) $z_c = \sqrt{\dfrac{1-\eta^2}{2(1+\eta^2)}}$, which is defined in Eq. (f7),

(iii) $w_0 = 2\eta\sqrt{\dfrac{gd}{1-\eta^2}}$ and $w_1 = 2\sqrt{\dfrac{d}{(1-\eta^2)g}}$,

In terms of y, w, and z_c,

$$a_+ = \frac{qV}{md} - g = g(2y^2-1) \qquad \text{(f18)}$$

$$a_- = \frac{qV}{md} + g = g(2y^2+1) \qquad \text{(f19)}$$

$$\mathrm{v}_{0+} = \mathrm{v}_s = w_0\sqrt{y^2+z_c^2} \qquad \text{(f20)}$$

$$\mathrm{v}_{0-} = \eta(\mathrm{v}_s + a_+t_+) = w_0\sqrt{y^2-z_c^2} \qquad \text{(f21)}$$

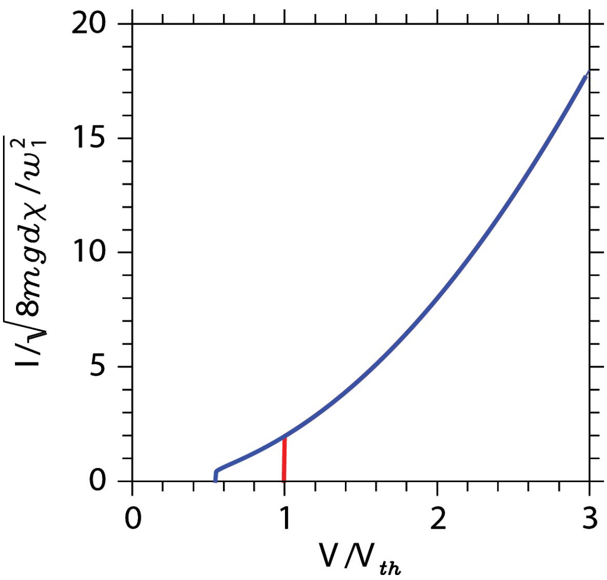
$$t_+ = w_1\frac{\sqrt{y^2-z_c^2}-\eta\sqrt{y^2+z_c^2}}{2y^2-1} \qquad \text{(f22)}$$

$$t_- = w_1\frac{\sqrt{y^2+z_c^2}-\eta\sqrt{y^2-z_c^2}}{2y^2+1} \qquad \text{(f21)}$$

$$I = \frac{\Delta Q}{\Delta t} = \frac{2q}{t_+ + t_-} = \left(2\chi V_{\text{th}}\right)\frac{y}{\Delta t} = \frac{\sqrt{8mgd\chi}}{w_1}F(y) \tag{f22}$$

where

$$F(y) = y\left\{\frac{\sqrt{y^2 - z_c^2} - \eta\sqrt{y^2 + z_c^2}}{2y^2 - 1} + \frac{\sqrt{y^2 + z_c^2} - \eta\sqrt{y^2 - z_c^2}}{2y^2 + 1}\right\}^{-1} \tag{f23}$$



3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents	
(a)	1.2	0.3	Gauss law, or a formula for the capacitance of a parallel plate	
		0.5	Total energy of a capacitor at V	E' = electrical field by the other plate
		0.4	Force from the energy expression	$F = QE'$
(b)	0.8	0.3	Gauss law	Use of area ratio and result of (a)
		0.5	Correct answer	
(c)	0.5	0.1	Correct lift-up condition with force balance	
		0.2	Use of area ratio and result of (a)	
		0.2	Correct answer	
(d)	2.3	0.5	Energy conservation and the work done by the field	
		0.5	Loss of energy due to collisions	
		0.8	Condition for the steady state: energy balance equation (loss = gain)	Condition for the steady state: recursion relation
		0.5	Correct answer	
(e)	2.2	0.2	$\Delta Q = 2q$ per trip	
		0.5	Acceleration $a_\pm$ in the limit of $qV >> mgd$; $a_+ = a_-$ by symmetry	
		0.3	Kinetic equations for d, v, a, and t, solutions for t_+	By using the symmetry, derive the relation (e4)
		0.4	Expression of $v_{0\pm}$ and $t_\pm$ in its steady state	
		0.4	Solutions of $t_\pm$ in approximation	
		0.4	Correct answer	
(f)	3.0	0.5	Condition for V_c; $K_{up} = 0$ or $v_{s,up} = 0$	Using (d8), Recursion relations
		0.3	energy balance equation	
		0.3	Correct answer of V_c	
		0.7	Kinetic equations for Δt	
		0.3	Correct answer of I_c	
		0.9	Distinction between V_{th} and V_c, the asymptotic behavior $I = \gamma V^2$ in plots	
Total	10			

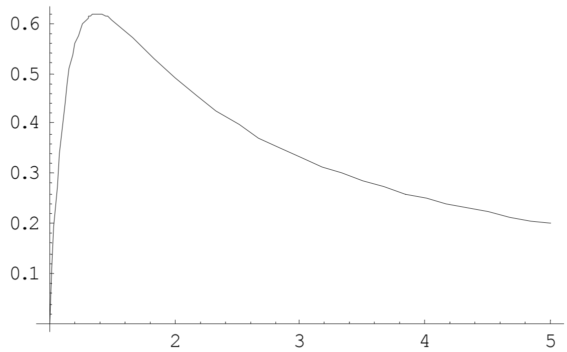
Theoretical Question 2: *Rising Balloon*

1. Answers

(a) $F_B = M_A ng \frac{P}{P+\Delta P}$

(b) $\gamma = \frac{\rho_0 z_0 g}{P_0} = 5.5$

(c) $\Delta P = \frac{4\kappa RT}{r_0}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right)$



(d) a =0.110

(e) z_f =11 km, λ_f =2.1.

2. Solutions

[Part A]

(a) [1.5 points]

Using the ideal gas equation of state, the volume of the helium gas of n moles at pressure $P+\Delta P$ and temperature T is

$$V = nRT/(P+\Delta P) \tag{a1}$$

while the volume of n' moles of air gas at pressure P and temperature T is

$$V = n'RT/P. \tag{a2}$$

Thus the balloon displaces $n' = n\dfrac{P}{P+\Delta P}$ moles of air whose weight is $M_A n' g$.

This displaced air weight is the buoyant force, i.e.,

$$F_B = M_A ng\frac{P}{P+\Delta P}. \tag{a3}$$

(Partial credits for subtracting the gas weight.)

(b) [2 points]

The pressure difference arising from a height difference of z is $-\rho gz$ when the air density ρ is a constant. When it varies as a function of the height, we have

$$\frac{dP}{dz} = -\rho g = -\frac{\rho_0 T_0}{P_0}\frac{P}{T}g \tag{b1}$$

where the ideal gas law $\rho T/P =$ constant is used. Inserting Eq. (2.1) and $T/T_0 = 1 - z/z_0$ on both sides of Eq. (b1), and comparing the two, one gets

$$\gamma = \frac{\rho_0 z_0 g}{P_0} = \frac{1.16\times 4.9\times 10^4 \times 9.8}{1.01\times 10^5} = 5.52. \tag{b2}$$

The required numerical value is 5.5.

[Part B]

(c) [2 points]

The work needed to increase the radius from r to $r+dr$ under the pressure difference ΔP is

$$dW = 4\pi r^2 \Delta P dr, \tag{c1}$$

while the increase of the elastic energy for the same change of r is

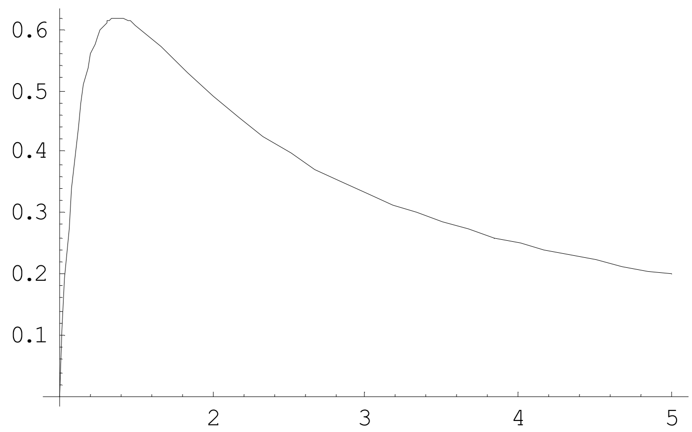
$$dW = \left(\frac{dU}{dr}\right)dr = 4\pi\kappa RT(4r - 4\frac{r_0^6}{r^5})dr\,. \tag{c2}$$

Equating the two expressions of dW, one gets

$$\Delta P = 4\kappa RT(\frac{1}{r} - \frac{r_0^6}{r^7}) \;=\; \frac{4\kappa RT}{r_0}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right). \tag{c3}$$

This is the required answer.

The graph as a function of λ (>1) increases sharply initially, has a maximum at $\lambda = 7^{1/6}$ =1.38, and decreases as λ^{-1} for large λ. The plot of $\Delta P/(4\kappa RT/r_0)$ is given below.



(d) [1.5 points]

From the ideal gas law,

$$P_0 V_0 = n_0 R T_0 \tag{d1}$$

where V_0 is the unstretched volume.

At volume $V = \lambda^3 V_0$ containing n moles, the ideal gas law applied to the gas inside at $T = T_0$ gives the inside pressure P_{in} as

$$P_{\text{in}} = nRT_0 / V = \frac{n}{n_0\lambda^3}P_0 \quad . \tag{d2}$$

On the other hand, the result of (c) at $T = T_0$ gives

$$P_{\text{in}} = P_0 + \Delta P = P_0 + \frac{4\kappa RT_0}{r_0}(\frac{1}{\lambda} - \frac{1}{\lambda^7}) = (1 + a(\frac{1}{\lambda} - \frac{1}{\lambda^7}))P_0\,. \tag{d3}$$

Equating (d2) and (d3) to solve for a,

$$a = \frac{n/(n_0\lambda^3) - 1}{\lambda^{-1} - \lambda^{-7}} . \tag{d5}$$

Inserting n/n_0=3.6 and λ=1.5 here, a=0.110.

[Part C]

(e) [3 points]

The buoyant force derived in problem (a) should balance the total mass of M_{T}=1.12 kg. Thus, from Eq. (a3), at the weight balance,

$$\frac{P}{P+\Delta P} = \frac{M_{\mathrm{T}}}{M_A n} . \tag{e1}$$

On the other hand, applying again the ideal gas law to the helium gas inside of volume $V = \frac{4}{3}\pi r^3 = \lambda^3 \frac{4}{3}\pi r_0{}^3 = \lambda^3 V_0$, for arbitrary ambient P and T, one has

$$(P+\Delta P)\lambda^3 = \frac{nRT}{V_0} = P_0 \frac{T}{T_0}\frac{n}{n_0} \tag{e2}$$

for n moles of helium. Eqs. (c3), (e1), and (e2) determine the three unknowns P, ΔP, and λ as a function of T and other parameters. Using Eq. (e2) in Eq. (e1), one has an alternative condition for the weight balance as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{M_{\mathrm{T}}}{M_A n_0} . \tag{e3}$$

Next using (c3) for ΔP in (e2), one has

$$P\lambda^3 + \frac{4\kappa RT}{r_0}\lambda^2(1-\lambda^{-6}) = P_0\frac{T}{T_0}\frac{n}{n_0}$$

or, rearranging it,

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{n}{n_0} - a\lambda^2(1-\lambda^{-6}) , \tag{e4}$$

where the definition of a has been used again.

Equating the right hand sides of Eqs. (e3) and (e4), one has the equation for λ as

$$\lambda^2(1-\lambda^{-6}) = \frac{1}{an_0}\left(n - \frac{M_{\mathrm{T}}}{M_A}\right) = 4.54. \tag{e5}$$

The solution for λ can be obtained by

$$\lambda^2 \approx 4.54/(1-4.54^{-3}) \approx 4.54 : \quad \lambda_f \cong 2.13. \tag{e6}$$

To find the height, replace $(P/P_0)/(T/T_0)$ on the left hand side of Eq. (e3) as a function of the height given in (b) as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = (1 - z_f / z_0)^{\gamma-1}\lambda_f^3 = \frac{M_{\mathrm{T}}}{M_A n_0} = 3.10\ . \qquad \text{(e7)}$$

Solution of Eq. (e7) for z_f with λ_f=2.13 and $\gamma - 1$=4.5 is

$$z_f = 49 \times \left(1 - (3.10/2.13^3)^{1/4.5}\right) = 10.9\ \text{(km)}. \qquad \text{(e8)}$$

The required answers are $\lambda_f = 2.1$, and $z_f = 11$ km.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.5	Archimedes' principle
		0.5	Ideal gas law applied correctly
		0.5	Correct answer (partial credits 0.3 for subtracting He weight)
(b)	2.0	0.8	Relation of pressure difference to air density
		0.5	Application of ideal gas law to convert the density into pressure
		0.5	Correct formula for γ
		0.2	Correct number in answer
(c)	2.0	0.7	Relation of mechanical work to elastic energy change
		0.3	Relation of pressure to force
		0.5	Correct answer in formula
		0.5	Correct sketch of the curve
(d)	1.5	0.3	Use of ideal gas law for the increased pressure inside
		0.4	Expression of inside pressure in terms of a at the given conditions
		0.5	Formula or correct expression for a
		0.3	Correct answer
(e)	3.0	0.3	Use of force balance as one condition to determine unknowns
		0.3	Ideal gas law applied to the gas as an independent condition to determine unknowns
		0.5	The condition to determine λ_f numerically
		0.7	Correct answer for λ_f
		0.5	The relation of z_f versus λ_f
		0.7	Correct answer for z_f
Total	10		

Theoretical Question 3: *Scanning Probe Microscope*

1. Answers

(a) $A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}$ and $\tan\phi = \dfrac{b\omega_0}{m(\omega_0^2 - \omega^2)}$. At $\omega = \omega_0$, $A = \dfrac{F_0}{b\omega_0}$

and $\phi = \dfrac{\pi}{2}$.

(b) A non-vanishing dc component exists only when $\omega = \omega_i$.

In this case the amplitude of the dc signal will be $\dfrac{1}{2} V_{i0} V_{R0} \cos\phi_i$.

(c) $\dfrac{c_1 c_2}{2} \dfrac{V_{R0}^2}{b\omega_0}$ at the resonance frequency ω_0.

(d) $\Delta m = 1.7 \times 10^{-18}$ kg.

(e) $\omega'_0 = \omega_0 \left(1 - \dfrac{c_3}{m\omega_0^2}\right)^{1/2}$.

(f) $d_0 = \left(k_e \dfrac{qQ}{m\omega_0 \Delta\omega_0}\right)^{1/3}$

$d_0 = 41$ nm.

2. Solutions

(a) [1.5 points]

Substituting $z(t) = A\sin(\omega t - \phi)$ in the equation $m\dfrac{d^2 z}{dt^2} + b\dfrac{dz}{dt} + m\omega_0^2 z = F_0 \sin\omega t$

yields,

$$-m\omega^2\sin(\omega t - \phi) + b\omega\cos(\omega t - \phi) + m\omega_0{}^2\sin(\omega t - \phi) = \frac{F_0}{A}\sin\omega t\,. \qquad \text{(a1)}$$

Collecting terms proportional to $\sin\omega t$ and $\cos\omega t$, one obtains

$$\left\{m(\omega_0^2 - \omega^2)\cos\phi + b\omega\sin\phi - \frac{F_0}{A}\right\}\sin\omega t + \left\{-m(\omega_0^2 - \omega^2)\sin\phi + b\omega\cos\phi\right\}\cos\omega t = 0 \qquad \text{(a2)}$$

Zeroing the each curly square bracket produces

$$\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}, \qquad \text{(a3)}$$

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}\,. \qquad \text{(a4)}$$

At $\omega = \omega_0$,

$$A = \frac{F_0}{b\omega_0} \quad \text{and} \quad \phi = \frac{\pi}{2}. \qquad \text{(a5)}$$

(b) [1 point]

The multiplied signal is

$$\begin{aligned} &V_{i0}\sin(\omega_i t - \phi_i)V_{R0}\sin(\omega t) \\ &= \frac{1}{2}V_{i0}V_{R0}[\cos\{(\omega_i - \omega)t - \phi_i\} - \cos\{(\omega_i + \omega)t - \phi_i\}] \end{aligned} \qquad \text{(b1)}$$

A non-vanishing dc component exists only when $\omega = \omega_i$. In this case the amplitude of the dc signal will be

$$\frac{1}{2}V_{i0}V_{R0}\cos\phi_i\,. \qquad \text{(b2)}$$

(c) [1.5 points]

Since the lock-in amplifier measures the ac signal of the same frequency with its reference signal, the frequency of the piezoelectric tube oscillation, the frequency of the cantilever, and the frequency of the photodiode detector should be same. The magnitude of the input signal at the resonance is

$$V_{i0} = c_2 \frac{F_0}{b\omega_0} = \frac{c_1 c_2 V_{R0}}{b\omega_0}. \qquad \text{(c1)}$$

Then, since the phase of the input signal is $-\frac{\pi}{2} + \frac{\pi}{2} = 0$ at the resonance, $\phi_i = 0$ and the lock-in amplifier signal is

$$\frac{1}{2} V_{i0} V_{R0} \cos 0 = \frac{c_1 c_2}{2} \frac{V_{R0}^2}{b\omega_0}. \qquad \text{(c2)}$$

(d) [2 points]

The original resonance frequency $\omega_0 = \sqrt{\frac{k}{m}}$ is shifted to

$$\sqrt{\frac{k}{m+\Delta m}} = \sqrt{\frac{k}{m}} \left(1 + \frac{\Delta m}{m}\right)^{-\frac{1}{2}} \cong \sqrt{\frac{k}{m}} \left(1 - \frac{1}{2}\frac{\Delta m}{m}\right) = \omega_0 \left(1 - \frac{1}{2}\frac{\Delta m}{m}\right). \qquad \text{(d1)}$$

Thus

$$\Delta\omega_0 = -\frac{1}{2}\omega_0 \frac{\Delta m}{m}. \qquad \text{(d2)}$$

Near the resonance, by substituting $\phi \to \frac{\pi}{2} + \Delta\phi$ and $\omega_0 \to \omega_0 + \Delta\omega_0$ in Eq. (a3), the change of the phase due to the small change of ω_0 (not the change of ω) is

$$\tan\left(\frac{\pi}{2} + \Delta\phi\right) = -\frac{1}{\tan\Delta\phi} = \frac{b}{2m\Delta\omega_0}. \qquad \text{(d3)}$$

Therefore,

$$\Delta\phi \approx \tan\Delta\phi = -\frac{2m\Delta\omega_0}{b}. \qquad \text{(d4)}$$

From Eqs. (d2) and (d4),

$$\Delta m = \frac{b}{\omega_0}\Delta\phi = \frac{10^3 \cdot 10^{-12}}{10^6} \frac{\pi}{1800} = \frac{\pi}{1.8} 10^{-18} = 1.7 \times 10^{-18} \text{ kg}. \qquad \text{(d5)}$$

(e) [1.5 points]

In the presence of interaction, the equation of motion near the new equilibrium position h_0 becomes

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+m\omega_0^2 z-c_3 z=F_0\sin\omega t \qquad \text{(e1)}$$

where we used $f(h)\approx f(h_0)+c_3 z$ with $z=h-h_0$ being the displacement from the new equilibrium position h_0. Note that the constant term $f(h_0)$ is cancelled at the new equilibrium position.

Thus the original resonance frequency $\omega_0=\sqrt{\frac{k}{m}}$ will be shifted to

$$\omega'_0=\sqrt{\frac{k-c_3}{m}}=\sqrt{\frac{m\omega_0^2-c_3}{m}}=\omega_0\sqrt{1-\frac{c_3}{m\omega_0^2}}\,. \qquad \text{(e3)}$$

Hence the resonance frequency shift is given by

$$\Delta\omega_0=\omega_0\left[\sqrt{1-\frac{c_3}{m\omega_0^2}}-1\right]. \qquad \text{(e4)}$$

(f) [2.5 points]

The maximum shift occurs when the cantilever is on top of the charge, where the interacting force is given by

$$f(h)=k_e\frac{qQ}{h^2}\,. \qquad \text{(f1)}$$

From this,

$$c_3=\left.\frac{df}{dh}\right|_{h=d_0}=-2k_e\frac{qQ}{d_0^3}\,. \qquad \text{(f2)}$$

Since $\Delta\omega_0<<\omega_0$, we can approximate Eq. (e4) as

$$\Delta\omega_0\approx-\frac{c_3}{2m\omega_0}\,. \qquad \text{(f3)}$$

From Eqs. (f2) and (f3), we have

$$\Delta\omega_0=-\frac{1}{2m\omega_0}\left(-2k_e\frac{qQ}{d_0^3}\right)=k_e\frac{qQ}{m\omega_0 d_0^3}\,. \qquad \text{(f4)}$$

Here $q=e=-1.6\times10^{-19}$ Coulomb and $Q=6e=-9.6\times10^{-19}$ Coulomb. Using the values provided,

$$d_0 = \left(k_e \frac{qQ}{m\omega_0 \Delta\omega_0} \right)^{1/3} = 4.1 \times 10^{-8} \text{ m} = 41 \text{ nm}. \qquad \text{(f5)}$$

Thus the trapped electron is 41 nm from the cantilever.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.7	Equations for A and ϕ (substitution and manipulation)
		0.4	Correct answers for A and ϕ
		0.4	A and ϕ at ω_0
(b)	1.0	0.4	Equation for the multiplied signal
		0.3	Condition for the non-vanishing dc output
		0.3	Correct answer for the dc output
(c)	1.5	0.6	Relation between V_i and V_R
		0.4	Condition for the maximum dc output
		0.5	Correct answer for the magnitude of dc output
(d)	2.0	0.5	Relation between Δm and $\Delta\omega_0$
		1.0	Relations between $\Delta\omega_0$ (or Δm) and $\Delta\phi$
		0.5	Correct answer (Partial credit of 0.2 for the wrong sign.)
(e)	1.5	1.0	Modification of the equation with $f(h)$ and use of a proper approximation for the equation
		0.5	Correct answer
(f)	2.5	0.5	Use of a correct formula of Coulomb force
		0.3	Evaluation of c_3
		0.6	Use of the result in (e) for either $\Delta\omega_0$ or $\omega'^2_0 - \omega_0^2$
		0.6	Expression for d_0
		0.5	Correct answer
Total	10		